MASTERY OF BASIC NUMBER COMBINATIONS: INTERNALIZATION OF RELATIONSHIPS OR FACTS?

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The thesis is advanced that children do not learn and store basic number combinations as so many separate entities or bonds (as hundreds of specific numerical associations) but as a system of rules, procedures, and principles as well as arbitrary associations. In this view, "mastering the basic facts" largely involves discovering, labeling, and internalizing relationships—processes encouraged by teaching thinking strategies. Moreover, internalized rules, procedures, and principles may become routinized and may help to account for the efficient production of number combinations in adults. Given an infinitely large arithmetic system, the use of such automatic reconstructive processes would make sense—would be cognitively economical. Accessibility, which has been advanced to account for anomalous retrieval time results, could be affected by input from semantic and procedural knowledge.

Accurate and automatic production of the basic number combinations is a major objective of elementary mathematics education. Typically, it is not an objective that is easily and quickly attained. Indeed, teachers regularly lament about how much difficulty children have in mastering the basic "number facts." This "problem" may be due, in part, to educators' misconceptions of (a) how children learn the basic number combinations and (b) how number combinations are represented in adult long-term memory. First this paper outlines the historical debate on how number combinations are learned or internalized. Then it critically reviews the empirical evidence for, and conceptual adequacy of, current models of how number combinations are represented in, and efficiently produced from, long-term memory. An alternative view is offered that argues that, though adults may automatically recall

1 Basic number combinations will refer to the 100 addition combinations with single-digit addends (0 + 0 to 9 + 9) and the 21 combinations in the series 10 + 0 to 10 + 10 (including their commuted pairs). It will also refer to the corresponding subtraction, multiplication, and division combinations. In general, the term number combinations rather than number facts will be used. Number fact connotes a mechanical or rote associative process and will be used to denote that meaning. Number combinations may be learned in a meaningful manner, and I prefer this less prejudicial term (cf. Brownell, 1935).
specific numerical associations between two digits and, say, their sum (a *reproductive* process), many combinations can be accurately and automatically produced from stored rules, procedures, or principles (efficient *reconstructive* processes).

**VIEWS ON MASTERING THE NUMBER COMBINATIONS**

One view of arithmetic learning that arose early in this century was the "drill theory," a product of associative theories of learning. This theory assumed that (a) children must learn to imitate the skills and knowledge of adults; (b) what is learned are associations or bonds between otherwise unrelated stimuli; (c) understanding is not necessary for the formation of such bonds; and (d) the most efficient way to accomplish bond formation is drill (Brownell, 1935). Because drill theory assumed that adults retrieved number facts from associative memory, the goal and method of instruction were clear. Children must form and strengthen bonds between two digits and, for addition, their sum (Thorndike, 1922). Such links or associations are strengthened largely by means of repetition.

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Drill theory proponents did not consider as important vehicles for mastering the number facts, children's counting strategies, discovery of relationships, or invented procedures (devices for reasoning out sums, differences, etc.). Indeed, counting and invented procedures were viewed as hindrances—as attempts to evade the real work of “memorizing” the number facts. For example, Wheeler (1939) explained the relative difficulty of large number facts this way:

As the size of the addend seems to be the general factor in causing the differences in the difficulty ranking, we wonder if the children are not computing the sums by physical or mental *counting*, a *crutch* which is probably developed in the child while building the number concepts [italics added]. Psychologically the child should be able to learn $5 + 4 = 9$ as easily as $2 + 3 = 5$. (p. 311)

Similarly, Smith (1921) considered invented procedures an impediment to learning the facts:

Another pupil required a long time for the sum of 6 and 9. He explained his process as follows: "6 and 10 are 16; 6 and 9 are 1 less than 6 and 10; then 6 and 9 are 15." He had to think through a similar form every time any number was added to 9 and of course gave much slower responses. . . . We should be careful about letting pupils acquire forms or roundabout schemes for

2Thorndike's (1922) association theory of number fact learning was actually more sophisticated than this basic model. He argued that frequency of practice was not sufficient to account for number fact learning. He argued that bonds should not be formed independently—that instruction should be organized so as to build on earlier, related learning. In addition to readiness, Thorndike argued that internal factors such as interest play a role in learning the number facts. Moreover, rather than recommend forming bonds for each individual fact, he appeared to advocate the learning of rules—albeit in other terms. For example, he noted that "the facts are best learned once for all as the habits '1 times k is the same as k' and 'k times 1 is the same as k'" (pp. 144–145). It is not clear, however, whether he was also advocating the use of the commutativity principle or just the learning of two rules (one for $N \times 1$ and another for $1 \times N$).
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securing a result in the lower grades which will prove a handicap to them in the upper grades. (pp. 764–765)

In sum, drill theory proposed that adult number fact knowledge (i.e., a network of automatic associations) is best achieved directly—by drill.

Early in this century, another—very different—account of arithmetic learning was advanced (see, for example, Brownell & Chazal, 1935; Buswell & Judd, 1925). According to Brownell (1935):

The “meaning” theory conceives arithmetic as a closely knit system of understandable ideas, principles, and processes. According to this theory, the test of learning is not mere mechanical facility in “figuring.” The true test is an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significance. (p. 19)

In this view, drill, counting, discovery of relationships, and use of invented procedures each have their place in learning the number combinations. Brownell (1935) noted that, initially, counting may be a necessary arithmetic strategy for children because it may be their only means of dealing with numbers. As soon as they are ready, however, children should be introduced to more mature methods. Brownell suggested, for example, the procedure of transforming an unfamiliar problem into a familiar one (e.g., 7 + 5: [7 + 3] + [5 – 3] = 10 + 2 = 12). Eventually, the child “comes to a confident knowledge of [a number combination], a knowledge full of meaning because of its frequent verification. By this time, the difficult stages of learning will long since have been passed, and habituation occurs rapidly and easily” (p. 24). Drill may serve to increase the facility and permanence of recall.

Meaning theory differs from drill theory in its view of number combination learning, then, in several important respects. First, learning mathematics—including mastery of the number combinations—is viewed as a slow, protracted process. Children are not expected by meaning theory to imitate immediately the skill or knowledge of adults. In other words, the child’s psychological readiness for learning is considered. At first, children are expected to engage in immature strategies (the use of counting and later invented procedures). Mature knowledge (including a mastery of number combinations and an understanding of mathematical principles) then evolves from this experience. In sum, “children attain ‘mastery’ only after a period during which they deal with combinations by procedures less advanced (but to them more meaningful) than automatic responses” (Brownell, 1941, p. 96). Second, because it arises from meaningful experience, habitual production of the number combinations has underlying meaning.

Research tends to support meaning theory. Counting experience is now viewed as an important basis for understanding arithmetic and performing it mentally (e.g., Gelman & Gallistel, 1978; Ginsburg, 1982; Resnick, 1983; Steffe, von Glasersfeld, Richards, & Cobb, 1983). Invented procedures and other rules or principles are frequently advocated as aids in mastering the basic combinations (e.g., Cobb, 1983; Folsom, 1975; Rathmell, 1978;
some research (e.g., Brownell & Chazal, 1935; Steinberg, 1984; Swenson, 1949; Thiele, 1938; Thornton, 1978) indicates that teaching children “thinking strategies” is more effective than drill in facilitating learning, retention, and transfer of basic combinations (Suydam & Weaver, 1975). Olander (1931) found that children who studied only 55 basic addition or subtraction combinations had the same level of gain in mastery as children who studied 100 combinations for each operation; the first group of children transferred their learning almost completely to the 45 untaught combinations for each process. Apparently the children in the first group did not simply learn isolated numerical associations but relationships or procedures that they could generalize to new problems.

Yet many educators still believe that learning the basic number combinations is essentially a straightforward, rote memory task that should be accomplished quickly. Indeed, curriculum guides frequently overestimate how quickly children should master the combinations. For example, Mathematics K–6: A Recommended Program for Elementary Schools (University of the State of New York & New York State Education Department, 1980) includes mastery of the addition and subtraction facts (sums and minuends to 18) as an objective for the second grade. (The third-grade objective is mastery of the addition and subtraction facts to 25.) Such guidelines overlook the psychological evidence that mastery of basic addition and subtraction is often not achieved until third grade or even later (e.g., Ashcraft, 1982; Carpenter & Moser, 1983, 1984; Lankford, 1974; Woods, Resnick, & Groen, 1975) and that all families of basic addition or subtraction combinations are not mastered in one year’s time (e.g., Baroody, 1983; Ginsburg & Baroody, 1983). Duckworth (1982) wisely points out that most things worth knowing take a long time to learn and that teacher training needs to reinforce this point. Such advice is appropriate even when we consider the teaching and learning of basic skills such as the efficient production of number combinations.

VIEWS ON MENTAL REPRESENTATION

Today it is commonly assumed that, over the course of development, children replace slow counting procedures and thinking strategies (inefficient reconstructive processes) with rapid fact retrieval (an efficient reproductive process) in order to do simple mental arithmetic (e.g., Ashcraft, 1982; Ilg & Ames, 1951; Resnick & Ford, 1981). Though a number of models (e.g., Ashcraft, 1982; Siegler & Shrager, 1984) have been proposed for how the number combinations are organized in an adult’s memory, all share the assumption that some kind of reproductive process underlies production of the basic facts. Winkelman and Schmidt (1974) propose that addition and multiplication facts share a parallel organization. They argue that there are associations between two digits (e.g., 3 + 3) and both their sum (6) and product (9). As a result of associative interference, there is a greater tendency
to associate 9 with 3 + 3 than, say, 7. Another model (Ashcraft & Battaglia, 1978; Ashcraft & Stazyk, 1981) proposes that the addition facts are mentally represented in memory as a printed table. The time needed to produce a particular fact is determined, in part, by the mental "distance" traversed during a memory search (i.e., the time needed to find the intersection of the two addends in the table). This helps to account for the slightly longer reaction times for problems with larger addends. Groen and Parkman (1972) suggest a direct-access model in which all remembered facts are equally accessible. Those facts that are not committed to memory are generated by the more immature counting-on (reconstructive) strategy. Resnick and Ford (1981) give the following example to illustrate this model:

If an adult were asked "How much is 3 + 4?" he or she would probably know immediately without really having to figure out the answer. Most adults have stored in long-term memory a response, 7, that is linked with the stimulus 3 + 4. It is as if there is a huge directory in their heads, and some of the entries were number facts that merely had to be "looked up." But think, now, what happens when a person has an occasional lapse of memory, when a number fact slips out of grasp. The answer is usually reconstructed in some way. (p. 74)

In brief, according to current theories, efficient production of number combinations is exclusively a reproductive process. Reconstructive processes are sometimes viewed as a less efficient back-up when a basic combination is not committed to long-term memory.

A number of chronometric studies have been undertaken, but no single model of mental representation has emerged as superior. In studies with efficient producers of number combinations, differences among the models are rarely, if ever, significant (Kevin Miller, personal communication, 22 June 1983). Moreover, though Ashcraft (1982) argues that his empirical evidence supports a tablelike fact retrieval (sum-squared) model, the case for this (or any other association-based) model is not entirely convincing (Baroody, 1984). Ashcraft used a verification task (the subject is presented with an equation such as 5 + 3 = 9 and is required to respond true or false) to generate his data. However, the verification task produces different results with older subjects than does a more straightforward production task (the subject is presented with a problem such as 5 + 3 and required to produce the answer). More specifically, combinations with 0 tend to be verified inefficiently but produced efficiently. Indeed, the reaction time latency curves for verification and production data typically differ in shape. It appears, then, that because of the extra decision stages required by the verification process, verification data may not accurately reflect memory search times and may not be the most suitable basis for drawing conclusions about the mental representation of number combinations.

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3That is, structural variables such as sum (a counting-all model), min (a counting-on from larger addend model), and sum squared (a tablelike retrieval model in which the mental distance between larger numbers becomes increasingly stretched) seldom significantly stand out as predictors in multiple regression analyses of chronometric data.
Ashcraft, Fierman, and Bartolotta (1984) attempted to address the latter criticism by reporting data that apparently show that the verification and production tasks yield compatible results (parallel functions). The primary analysis (an analysis of variance) involved the factors of grade, task, and problem size. Unfortunately, problem size was analyzed in terms of a relatively crude small-sum versus large-sum breakdown. No justification was given for collapsing the data in a manner that might mask differences in the tasks.

Ashcraft et al. (1984) also note that, though the slopes for their verification and production data are not parallel for previously used predictors (models), they are parallel when a new predictor (a revised retrieval model) is used. The revised model includes accessibility as a basic factor in retrieval of number combinations. Ashcraft et al. argue that accessibility is related to memory-trace strength factors such as acquisition readiness, frequency of practice, and recency of use. Accessibility has been proposed to account for the exceptions to simple number-size rules (e.g., reaction time increases as the size of the addends increases) (Campbell & Graham, in press). Thus accessibility permits association models to distinguish between information location (determined by number size) and the ease of getting the information (affected by other learning and memory variables) (Miller, Perlmutter, & Keating, 1984).

To gauge this accessibility, Ashcraft proposes the use of a difficulty index. The more difficult the problem, the lower its accessibility score; the less difficult, the higher the accessibility score. One difficulty index measure was the subjects' subjective ratings of problem difficulty. A second index was the Wheeler (1939) difficulty ratings. With the Wheeler variable as the predictor variable, the slopes for the verification and production data are indeed parallel.

However, it is not clear why the use of the Wheeler index is justified. Apparently, the Wheeler difficulty norms were based on the number of second graders who “mastered” each fact. Unfortunately, the criterion of success was not precisely defined. Moreover, the subjects in the Wheeler study were given a particular type of training. Though the Wheeler index correlated well with a number of other difficulty indices of its day, there is no reason to believe it was or is a definitive measure of difficulty. After all, such indices are affected by the nature of the task, age of the subjects, criterion of success, previous training of the subjects, and so forth. Unfortunately, data on the subjects’ subjective difficulty ratings, which would seem a more appropriate index, were not reported by Ashcraft et al. (1984). In brief, it still does not appear safe to assume that the verification data accurately reflect differences in the search and retrieval time of number combinations.

Clearly, one barrier in finding a clearly superior model of number-combination representation and retrieval is the limitations of current methodology. First, production and verification data are “noisy.” Both tasks assume a more or less unidirectional sequence of stages or processes (e.g., see
Ashcraft, 1982). Yet solving even simple problems may not involve a straightforward sequence of steps. For example, a solution (or partial solution) may be checked for plausibility. An "out of limits" solution may cause the solution routine (subroutine) to restart. This may be especially true in ambiguous situations such as the verification task where a subject is given a problem like $7 + 3 = 21$. Indeed, because it introduces ambiguity and because it involves an extra decision stage, the verification task may be especially noisy. Second, the aim of chronometric analyses has been to uncover the way in which adults generate number combinations (cf. Siegler & Robinson, 1982). Yet detailed observation of adult arithmetic performance suggests that adults (like children) use a variety of strategies (see, e.g., Browne, 1906). In effect, adult chronometric data may reflect an averaging over different strategies rather than the use of a single process (cf. Siegler & Robinson, 1982), and this may help to account for inconsistencies within the data of a single subject or sample and the inconsistencies among chronometric studies (cf. Baroody, 1984). Third, chronometric data cannot differentiate between input from long-term memory and working memory and thus cannot definitively account for the cause of production (or verification) errors. Associative interference (the tendency to incorrectly give the answer of a related or previous problem) has been taken as clear evidence that the basic number combinations are stored in an associative network (e.g., Ashcraft, 1982; Campbell & Graham, in press; Winkelman & Schmidt, 1974). However, it is entirely possible that, when given a series of related problems, a person might hold previously generated combinations in working memory to facilitate further problem solving (cf. Baroody, Ginsburg, & Waxman, 1983). Input from working memory may also interfere with accurately generating later solutions. Interference is especially likely to come from combinations most recently generated and from difficult combinations requiring more checking.

Moreover, it may be that a clearly superior model has not emerged because current models are somehow inaccurate or incomplete. That is, the mental representation and efficient recall of number combinations may be more elaborate than simple associative models allow (Baroody, 1983). Because people are flexible information processors, they may use several means—including reconstructive processes—to efficiently generate number combinations (Baroody & Ginsburg, 1982). It may be that some combinations are generated quite quickly from stored invented procedures. Many others may be produced rapidly and directly from rules or principles that form an adult's mathematical semantic (meaning) system. Exploiting already internalized regularities or relationships eliminates the need to learn and store hundreds of individual numerical associations. In sum, using stored procedures, rules, or principles to quickly construct a range of combinations is cognitively more economical than relying exclusively on a network of individually stored facts. Take, for example, the $(N + 0$ and $0 + N)$ family of combinations: $1 + 0$, $2 + 0$, $10 + 0$ and $0 + 1$, $0 + 2$, $10 + 0$. I worked with one
kindergarten girl who was puzzled by $6 + 0$ and finally responded “60.” After I helped her see that 0 was the number name for adding nothing, she answered $6 + 0$ correctly. Later in the session, she immediately responded to $3 + 0$ with “3.” A week later—without further intervention—she correctly and automatically responded to $0 + 5$, $3 + 0$, $4 + 0$, $7 + 0$, $6 + 0$, and $0 + 8$ (combinations without 0 were interspersed). It appears that the child initially had an informal identity rule: “Nothing added to a set does not change the set.” When she was introduced to the term “zero” and the written symbol 0, she assimilated this formal mathematics in terms of the informal identity rule. The result was a formal $(N + 0$ and $0 + N = N)$ rule: “When zero and a number are added, the sum is the number.” By using this abstracted rule, the child then appeared able to answer any $(N + 0)$ or $(0 + N)$ problem quickly and accurately (cf. Miller & Wellman, 1984; Thiele, 1938). How else can the transfer to the $3 + 0$, $0 + 5$, and other combinations be explained? In all probability, she had never been exposed to—let alone practiced—the combination “$3 + 0$ is 3” or “$0 + 5$ is 5.” Thus it is not clear how association/fact-retrieval models can account for such behavior.

In contrast to current models that posit associations among specific numbers, then, the alternative model allows that efficient generation of number combinations is due, in part, to storing and using algebraic or verbal labels. Thus instead of forming and storing individual associations for $3 + 0$ and 3, $0 + 5$ and $5$, $88 + 0$ and $88$, and $0 + 1 000 000 000$ and $1 000 000 000$, and so on, the child may abstract a relationship and summarize the relationship in algebraic terms $(N + 0$ and $0 + N = N)$ or use a (verbal) rule (“When zero and a number are added, the sum is the number”). Then when new problems are introduced, the most relevant algebraic expression or label is sought and used to reconstruct an answer. Such an economical process makes sense with an infinitely large number system.

In the case of combinations with 0 $(N + 0 = N$, $0 + N = N$, $N - 0 = N$, $N \times 0 = 0$, and $0 \times N = 0$), the algebraic or verbal rules are relatively easy to learn. This would help to account for the observations that combinations with 0 are mastered relatively early (e.g., Groen & Parkman, 1972; Woods, Resnick, & Groen, 1975). Yet because of the form in which they are stored, the combinations with 0 may be particularly susceptible to performance failures. That is, because the algebraic or verbal rules are so similar, they are relatively easy to confuse. Though he referred to “designs” rather than rules, Thyne (1954) observed 30 years ago that “these designs would seem to be very ‘easy’ to learn—a possibility which might be expressed paradoxically by saying that it is the very ‘easiness’ of the zero facts which makes them so ‘difficult.’ In other words, even young pupils can soon acquire a knowledge of how to answer zero facts in a way which is at once most consistent and most unreliable” (p. 205). Indeed, confusion in selecting among the rules is especially likely to occur in verification situations, where the stimulus (e.g.,
$5 \times 0 = 5$) may trigger two conflicting rules ($N \times 0 = 0$ and $N + 0 = N$).

More recently, Ashcraft (1983) has allowed that reconstructive processes may play some role in the efficient production of number combinations but that this role is limited to the special cases involving 0 and 1. Specifically, he is willing to grant that children may learn ($N + 0$ and $0 + N = N$) and ($N - 0 = N$) rules and use well-learned “just after” or “just before” count sequence relationships to generate solutions to ($N + 1$) and ($N - 1$) problems. But there are other possibilities for the rule-governed production of basic combinations. Consider the basic subtraction combinations. Children may quickly learn an ($N - N = 0$) rule or identity principle to efficiently deal with combinations such as $2 - 2$, $9 - 9$, $86 - 86$. For combinations with terms that differ by 1 (e.g., $6 - 5$, $7 - 6$, $8 - 7$, or even $106 - 105$), the child might realize that the answer is always 1 (“The subtraction of ‘number neighbors’ produces a difference of one”). Some adults may continue to use a nine rule: “A teen $N$ minus 9 is $N + 1$” (e.g., $16 - 9 = 7$, $17 - 9 = 8$, $18 - 9 = 9$). Some subtraction combinations may be efficiently reconstructed from their addition counterparts (e.g., $10 - 7$ is 3 because $7 + 3$ is 10) (Baroody, 1983; Baroody et al., 1983; Carpenter & Moser, 1984; Steinberg, 1984). In terms of the other operations, various thinking strategies (e.g., see Carpenter & Moser, 1984; Folsom, 1975; Jerman, 1970; Rathmell, 1978; Trivett, 1980) may become routinized procedures for generating some addition and multiplication combinations. Ashcraft (1982) himself notes that some people may not store or only weakly store the basic division combinations because such combinations can be generated from knowledge of the multiplication combinations. In brief, efficient reconstructive processes may play a role in more than the efficient production of 0 and 1 combinations.

IMPLICATIONS FOR RESEARCHERS

The alternative model raises the possibility that a member of a combination family might be stored in the form of a specific numerical association and represented by a rule (R. S. Siegler, personal communication, 18 May 1984). For example, $1 + 0$ of the zero addition family might be stored as the fact $1 + 0 = 1$ and represented by the ($N + 0$ and $0 + N = N$) rule. Less familiar family members such as $0 + 81$ or $9077 + 0$ would probably be represented solely by the rule. Likewise, through repeated exposure, the child can learn the specific numerical association $1 + 1 = 2$. This combination could also be represented by the “$N$ after” rule (“adding 1 to an $N$ equals the number after $N$ in the count sequence”). Again, less familiar family members such as $1 + 81$ or $9077 + 1$ would probably be reconstructed by using the $N$-after rule in conjunction with the mental representation of the count sequence.

Such a proposal raises several questions. First, what is the extent of multiple representation? For instance, where does the multiple representation end for the ($N + 0$ and $0 + N = N$) family? With $N = 9$? Though all the basic combinations in this and other families might be represented both as facts and
by stored rules, procedures, or principles, it would not be cognitively economical to do so. It may be that only a few examples of a family are stored both as fact and rule. These examples might serve as focal instances—much as storing the image of, say, a collie might serve as an exemplar of a concept of “dog.” Second, if a combination is represented by both a fact and a rule, which representation is chosen or accessed? This might depend on the associative strength of a combination (cf. Siegler & Robinson, 1982; Siegler & Shrager, 1984). Very familiar combinations with great associative strengths might always be retrieved from a factual representation. The extent to which the factual representation of less familiar combinations is tapped might depend on the specific conditional probability of the fact and confidence criterion set by the subject. As noted earlier, unfamiliar combinations would probably be generated from the representation of the rule.

Another possibility is that reconstructive and reproductive processes work together to generate number combinations. Associative models have assumed a mechanical reproductive process that taps directly into associative memory, without input from semantic or procedural memory. The alternative model (Baroody, 1983, 1984) has implied that reproductive and reconstructive processes are parallel operations—each underlying the production of different families. However, knowledge of rules, procedures, or principles and specific numeral associations may, in various degrees, interact to generate number combinations (see Figure 1). For example, many subtraction combinations may be generated from knowledge of the complement principle and part-whole numerical relationships internalized as a result of learning addition combinations (Baroody, 1983; Baroody et al., 1983; Carpenter & Moser, 1984).

Indeed, existing evidence suggests that semantic factors are sometimes more important than problem size in determining how efficiently a number combination is produced. Svenson (1975) found that, by third grade, reaction times increased with addend size but dropped off with the \((10 + N \text{ and } N + 10)\) family. The relatively fast reaction times for the \((10 + N \text{ and } N + 10)\) family can probably be attributed to the emphasis in primary school on base 10 notions (e.g., a teen is a composite of a 10 and 1s). Campbell and Graham (in press) found evidence that semantic knowledge was taken into consideration in the generation of multiplication facts by adults. Specifically, a lower error rate for the \((N \times 5 \text{ and } 5 \times N)\) family was attributed to a rule that states, if a times problem contains 5, the product must end in 5 or 0—rendering implausible many of the products that would be associated with the other operands. Likewise, Miller et al. (1984) hypothesize that accessibility may vary because of the availability of redundant information, such as the knowledge that the product of two odd numbers will itself be odd. Such redundant information is presumably stored in semantic memory. Invoking accessibility to account for variations in retrieval error rate (e.g., Campbell & Graham, in press) and reaction time (Miller et al., 1984) sometimes appears to require
that the construct (and the association model) take into account input from semantic and procedural memory.

Because the alternative model suggests that more than a single mental process can account for efficient production of number combinations, more elaborate chronometric analyses or even new methodologies are needed to study the development and mental representation of number combinations. Because the rules, procedures, and principles that may underlie some combination families should vary in the ease in which they are learned and executed, chronometric analyses should be done by combination families. In addition, the role of semantic knowledge needs to be more systematically examined in families that extend beyond the basic single-digit combinations.

In any case, the study of number combination learning and representation needs to go beyond chronometric approaches. Because older children and adults produce answers so quickly and existing production measurement techniques are not sufficiently accurate, systematic differences in latencies may be masked (Groen & Parkman, 1972). Moreover, because some rule-governed or principled reconstructive processes may be as quick as a reproductive process, reaction time (and regression analysis) per se cannot reliably
distinguish between the two processes. Needed are longitudinal studies that carefully follow individual children's progress (including transfer) for various combination families from initial estimation performance to efficient production. Various brain research methods might fruitfully be used to study the issue of how number combinations are represented. For example, such techniques could indicate whether a good candidate for rule-governed production (e.g., $0 + 8$ or $76 + 0$) activates the same area of the brain as a good candidate for associative retrieval (e.g., $1 + 1$ or $5 + 5$). Though current efforts (e.g., Caramazza, McCloskey, & Basili, 1984) have not tested the alternative models directly, evaluation of brain-injured adults might be helpful as well. For example, if—as the alternative model suggests—knowledge of commutativity plays a role in the production of addition and multiplication combinations, then the pattern of competence or loss for commuted pairs (e.g., $7 + 5$ and $5 + 7$ or $7 \times 5$ and $5 \times 7$) should be similar.

**IMPLICATIONS FOR EDUCATORS**

If the alternative model is correct, children normally do not "memorize" and store all 400 or so basic combinations. In other words, children do not learn basic number combinations as so many separate entities or bonds (as hundreds of feats of memory) but as a system of interrelated experiences (Olander, 1931). Rules, procedures, and principles become routinized to make mastering the basic combinations a cognitively manageable task. This may help to explain why one of the most significant deficits displayed by children having difficulty with mathematics is weakness in their knowledge of basic number combinations (e.g., Russell & Ginsburg, 1984). Especially difficult for such children are larger combinations (e.g., Kraner, 1980; Smith, 1921). Because they do not have a rich network of rules and principles to invent or discover a heuristic for producing (larger) number combinations, such children are faced with the burden of memorizing many apparently isolated facts. Learning-disabled children and others experiencing mathematical difficulties may feel overwhelmed with such a chore and may give up trying to learn the basic combinations.

The teaching of thinking strategies has been advocated to help children (a) learn numerical relationships and (b) foster the automatic recall of number facts (e.g., Brownell, 1935). If the alternative model is correct, these two objectives are more closely related than commonly thought. The encouragement of thinking strategies may indeed help children to form specific numerical associations. Perhaps more importantly, it may also help make explicit and facilitate the internalization of rules, procedures, and principles that

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4 Brownell (1935) did not address directly the issue of how the number combinations were represented in long-term memory once they become automatic. He clearly implied that, because they arose from meaningful experience, habitual responses have underlying meaning. Not clear, though, is whether habitual responses implied mechanical retrieval (reproduction) from a network of specific numerical associations stored separately from semantic memory or efficient reconstruction drawing directly on information stored in semantic memory.
underlie whole families of combinations. Parenthetically, Cobb (1984) suggests that encouraging thinking strategies is important if for no other reason than the beliefs it fosters about mathematics. If we encourage the discovery and use of thinking strategies, children may be less likely to equate mathematics with arithmetic (knowing number facts and calculational routines) and more likely to appreciate the real nature of mathematics (the discovery and application of regularities and relationships).

This is not to say that regular practice is unimportant. Drill may not only foster the formation of specific numerical associations; it may also help routinize the application of rules, procedures, and principles. Thus drill is an important component of instruction once children have the opportunity to find relationships in order to facilitate internalization and the automatic use of such knowledge (cf. Brownell, 1935).

SUMMARY

In summary, in contrast to current models that view representation of the basic number combinations as a network of hundreds of specific numerical associations, it would seem cognitively more economical to mentally represent many groups or families of combinations in algebraic or verbal terms: as rules, procedures, or principles from which a whole range of combinations could be reconstructed. According to this alternative model, "mastery of the facts" would include discovering, labeling, and internalizing relationships. Meaningful instruction (the teaching of thinking strategies) would probably contribute more directly to this process than a drill approach alone. Thus, in contrast to current models that hypothesize reproductive processes replacing (slow) reconstructive processes, the alternative model suggests that some of the reconstructive processes involved in learning the combinations originally may continue to operate in adults, though more automatically. Indeed, internalized rules, procedures, and principles may interact with a network of specific numerical associations to account for efficient production of basic number combinations. Input from semantic and procedural knowledge would affect accessibility, a construct advanced to account for data that do not obey simple size-effect rules. In brief, the efficient production of basic number combinations does not seem to be a cognitively trivial process that can readily be promoted by unsophisticated educational practices.

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